Q1 (a) Find the equation for the normal vector, the tangent plane, and the normal line to the cone $z^2=x^2+y^2$ at the point where $x=3, y=4, \wedge z>0$

$z^2=x^2+y^2 z^2=9+16 z^2=\sqrt{25} \rightarrow z=5 P(3,4,5)$

$\nabla . f=\frac{\partial f}{\partial x} i+\frac{\partial f}{\partial y} j+\frac{\partial f}{\partial z} k$ ¿$2 xi+2 yj-2 zk$

Normal Vector: $\nabla f(p)=6 i+8 j-10 k$

$[(x-x^{\circ}) i+(y-y^{\circ}) j+(z-z^{\circ}) k] \bullet[6 i+8 j-10 k]=0$

$[(x-3) i+(y-4) j+(z-5) k] \bullet[6 i+8 j-10 k]=06(x-3)+8(y-4)-10(z-5)=0$

$6 x-18+8 y-24-10 z-50=0$

Tangent Plane: $6 x+8 y=10 z=92$

$[(x-x^{\circ}) i+(y-y^{\circ}) j+(z-z^{\circ}) k]=t[6 i+8 j-10 k]$

$[(x-3) i+(y-4) j+(z-5) k]=t[6 i+8 j-10 k]$

$x-3=6 t x=6 t+3 y-4=8 t y=8 t+4 z-5=-10 t z=-10 t+5$

Normal Line: $r(t)=[(6 t+3),(8 t+4),(-10+5)]$

Q1 (b) Let $f(x,y,z)=xyz$ and let $\boldsymbol{u}$ be a unit vector perpendicular to both vectors

$v=i-2j+k$

$w=2i+j-k$

Find the directional derivative of 'f 'at point P (1,-1, 2) in the direction of $\boldsymbol{u}$.

Solution:

$D_a f = \dfrac{a \bullet \nabla f}{|a|}$ , $a=u$

$grad f=\nabla f=\dfrac{\partial f}{\partial x}i+\dfrac{\partial f}{\partial y}j+\dfrac{\partial f}{\partial z}k$

$u=v\times w=\begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 2 & 1 & -1 \end{vmatrix}=i(2-1)-j(-1-2)+k(1+4)$

$u=i+3j+5k$

$\nabla f=yzi+xzj+xyk$

$\nabla f=-2i+2j-1k$

$|u|=\sqrt{35}$

$\dfrac{(i+3j+5k)}{\sqrt{35}}.(-2i+2j-k)=\dfrac{-1}{\sqrt{35}}$

Q1 (c) Find the divergence and the curl of the vector field:

$F=\left(z^2 e^{-x}\right) i+\left(y^3 lnz\right) j+\left(x e^{-y}\right) k$

Divergence $\nabla . V=\frac{d V_1}{dx}+\frac{d V_2}{dy}+\frac{d V_3}{dz}$

¿$-z^2 e^{-x}+3 y^2 lnz+0$

Curl $\nabla \times v=\begin{vmatrix} i & j & k \\ \frac{d V_1}{dx} & \frac{d V_2}{dy} & \frac{d V_3}{dz} \\ z^2 e^{-x} & y^3 lnz & x e^{-y} \end{vmatrix}$

¿$\left(\frac{dx e^{-y}}{dy}-\frac{d y^3 lnz}{dz}\right) i-\left(\frac{dx e^{-y}}{dx}-\frac{d z^2 e^{-x}}{dz}\right) j+\left(\frac{d y^3 lnz}{dx}-\frac{d z^2 e^{-x}}{dy}\right) k$

¿$\left(-yx e^{-y}-y^3 \frac{1}{z}\right) i-\left(e^{-y}-2 z e^{-x}\right) j+(0)$

¿$\left(-yx e^{-y}-\frac{y^3}{z}\right) i-\left(e^{-y}-2 z e^{-x}\right) j$

Q2 Evaluate the following integrals: $\iint_R xy$

a) $\iint_R x^2 e^{xy}\, dA$

$R: 0 \le x \le 1, 0 \le y \le 1$

$\int_0^1 \frac{x^2 e^{xy}}{x} \vee_0^1 dx$

¿ $\int_0^1 \left(x e^{xy}\right) \vee_0^1 dx$ ¿ $\int_0^1 (x e^x - x)\, dx$ I take x as a common factor

¿ $\int_0^1 x(¿ e^x - 1)\, dx$ ¿¿ $\int_0^1 x e^x dx - \int_0^1 x\, dx$

$u = x, dv = e^x dx du = dx\, v = e^x$

$udv = uv - \int vdu$

¿ $xe^x - \int_0^1 e^x dx - \int_0^1 xdx$ ¿ $\left(x e^x - e^x\right)_0$ $- \left(\frac{x^2}{2}\right)_0^1$ ¿ $(e - e) - (0 - 1) - \frac{1}{2} = \frac{1}{2}$

b) $\int_0^2 \int_0^{\sqrt{2x-x^2}} y\sqrt{x^2+y^2}\,dydx$

I find the limits in polar coordinates

$0 \le y \le \sqrt{2x-x^2} 0 \le y^2 \le 2x-x^2 y^2+x^2=2x x=r\cos\theta y=r\cos\theta \because r^2=x^2+y^2$

$\therefore r^2=2r\cos\theta r=2\cos\theta$

$0 \le r \le 2\cos\theta 0 \le \theta \le \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r\sin\theta r\,rdrd\theta$

$\int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^3 \sin\theta drd\theta$

$\int_0^{\frac{\pi}{2}} (\frac{r^4}{4})_0^{2\cos\theta} \sin\theta d\theta$

$\int_0^{\frac{\pi}{2}} 4\cos^4\theta\sin\theta d\theta$

Let $u=\cos\theta du=-\sin\theta d\theta$

$\int_1^0 -4u^4 du¿ \frac{4}{5}$

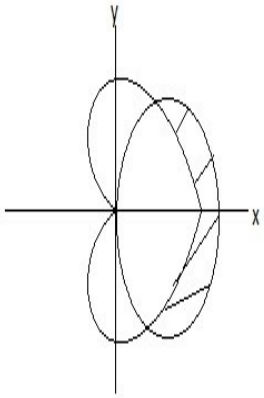
Q2 (c) Find the area of the region $\boldsymbol{R}$ that lies inside the circle $r=3\cos\theta$ and outside the cardioid $r=1+\cos\theta$

$r = 1+\cos\theta$

| r | $\theta$ |
|---|---|
| 2 | 0 |
| 1.86 | π/6 |
| 1.7 | π/4 |
| 1.5 | π/3 |
| 1 | π/2 |
| 0.5 | 2π/3 |
| 0.29 | 3π/4 |
| 0.13 | 5π/6 |
| 0 | π |
| 1 | −π/2 |
| 2 | 2π |

$r = 3\cos\theta$

| r | $\theta$ |
|---|---|
| 3 | 0 |
| 2.59 | π/6 |
| 2.12 | π/4 |
| 1.5 | π/3 |
| 0 | π/2 |
| -1.5 | 2π/3 |
| -2.12 | 3π/4 |
| -2.59 | 5π/6 |
| -3 | π |
| 0 | −π/2 |
| 3 | 2π |




$$\iint_{\frac{-\pi}{3}}^{\frac{\pi}{3}}\int_{1+\cos\theta}^{3\cos\theta} r\,dr\,d\theta$$

$$\int_{\frac{-\pi}{3}}^{\frac{\pi}{3}} \frac{r^2}{2}\Big|_{1+\cos\theta}^{3\cos\theta} d\theta = \frac{1}{2}\int_{\frac{-\pi}{3}}^{\frac{\pi}{3}} \text{¿¿¿}$$

$$\text{¿}\frac{1}{2}\int_{\frac{-\pi}{3}}^{\frac{\pi}{3}} 9\cos^2\theta-(1+2\cos\theta+\cos^2\theta)\,d\theta = \frac{1}{2}\int_{\frac{-\pi}{3}}^{\frac{\pi}{3}} 9\cos^2\theta-(1-2\cos\theta-\cos^2\theta)\,d\theta$$

$$\frac{1}{2}\int_{\frac{-\pi}{3}}^{\frac{\pi}{3}} 8\cos^2\theta-2\cos\theta-1\,d\theta$$

$$\text{¿}\frac{1}{2}\int_{\frac{-\pi}{3}}^{\frac{\pi}{3}} \frac{8}{2}(1+\cos 2\theta)-2\cos\theta-1\,d\theta$$

$$\frac{1}{2}\left(\frac{8}{2}\theta+\frac{8\sin 2\theta}{4}-2\sin\theta-\theta\right)\Bigg|_{\frac{-\pi}{3}}^{\frac{\pi}{3}}$$

¿ $\frac{1}{2}(\frac{8\pi}{6}+8\frac{\sqrt{3}}{8}-2\frac{\sqrt{3}}{2}-\frac{\pi}{3})$

¿ $\frac{1}{2}\left(\frac{8\pi}{6}+\sqrt{3}-\sqrt{3}-\frac{\pi}{3}\right)$

¿ $\left(\frac{8\pi}{12}-\frac{\pi}{6}\right)-\frac{1}{2}\left(\frac{-8}{6}-\frac{8\sqrt{3}}{8}+\sqrt{3}+\frac{\pi}{3}\right)$

¿ $\left(\frac{8\pi}{12}-\frac{\pi}{6}\right)+\left(\frac{8}{12}-\frac{\pi}{6}\right)$

¿ $\frac{16\pi}{12}-\frac{2\pi}{6}$

¿ $\frac{8\pi}{6}-\frac{2\pi}{6}=\frac{6\pi}{6}=\pi$

Q2 (d) Find the volume of the tetrahedron, $\boldsymbol{T}$ bounded by the plane $2x+y+3z=6$ and the coordinate planes $x=0, y=0, z=0$

The limits are

$$0 \le x \le 3, 0 \le y \le 6, 0 \le z \le 2$$

$$\int_0^3 \int_0^6 \int_0^2 dz\, dy\, dx$$

$$\int_0^3 \int_0^6 [z]_0^2 \, dy\, dx$$

$$\int_0^3 [2y]_0^6 \, dx$$

$$\int_0^3 12 \, dx$$

$$[12x]_0^3 = 36$$

Q3 Evaluate the line integral along C where $F(r)=[y, x+y]$ and C is the curve given by: $y=x^2+2x$ from (0, 0) to (2, 8)

Let x=1 $x=1$

$y=t^2+2t$

$r=[t, t^2+2t]$

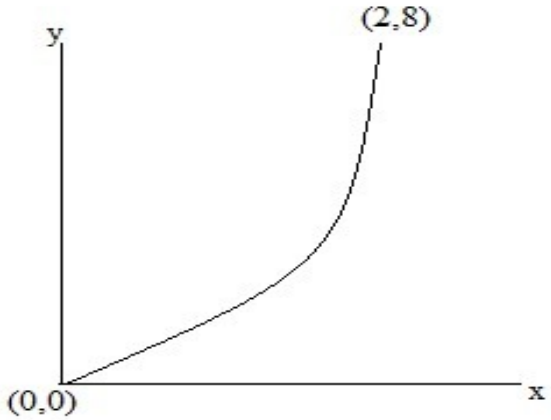
$F[r(t)]=[t^2+2t, t^2+3t]$

$r'(t)=[1, 2t+2]$

$F.dr=(t^2+2t)+(2t+2)(t^2+3t)$

¿$2t^3+9t^2+8t$

$$\int_c F.dr=\int_0^2 (2t^3+9t^2+8t)dt=48$$



4) Use Green's theorem to evaluate the line integral

$\oint_c [(x+y^2)dx+(1+x^2)dy]$ where C is the closed curve determined by: $y^2=x$ and $y=x^3$ with $0\le x\le 1$

Solution:

$$\oint_c F_1\,dx+F_2\,dy=\iint_R \left(\frac{dF_2}{dx}-¿\frac{dF_1}{dy}\right)dx\,dy¿$$

Limits:

$$0 \le x \le 1$$

$$x^3 \le y \le \sqrt{x}$$

$$=\int_{x^3}^{\sqrt{x}}\int (2x-2y)\,dy\,dx$$

$$=\int_0^1 \left((2x)(\sqrt{x})-\sqrt{x}^2\right)-\left((2x)(x^3)-x^6\right)dx$$

$$=\int_0^1 (2x^{\frac{3}{2}}-x-2x^4+x^6)\,dx$$

$$=\left(\frac{4}{5}x^{\frac{5}{2}}-\frac{x^2}{2}-\frac{2x^5}{5}+\frac{x^7}{7}\right)$$

Substitute 0 and 1

$$=\left(\frac{4}{5}\left(1^{\frac{5}{2}}\right)-\frac{1^2}{2}-\frac{2(1^5)}{5}+\frac{1^7}{7}\right)$$

$$=\frac{3}{70}$$

Q5 Evaluate the surface integral $\iint_S F.ndA$ for the vector $F=2i+5j+3k$;

where the surface S is the portion of the cone $z^2=x^2+y^2$ that lies inside the cylinder

$x^2+y^2=1$ and the planes z = 0 and z = 1. The parametric representation of a cone is given by: $r(r,\theta)=[r\cos\theta, r\sin\theta, r]$.

Solution:

Parametric representation of a cone:

$(x,y,z)(r\cos\theta, r\sin\theta, r)$

$N=\bar{r}_r\times\bar{r}_\theta$

$\bar{r}_r=(\cos\theta,\sin\theta,1)$

$\bar{r}_\theta=(-r\sin\theta, r\cos\theta, 0)$

$N=\bar{r}_r\times\bar{r}_\theta$

$$N=\begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 1 \\ r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$=[-r\cos\theta i, r\sin\theta, r]$$

$$F(r(r,\theta))=[2,5,3]$$

$$\text{F.N} = -2r\sin\theta-5\sin\theta+3\text{r}$$

$$\iint_S F.ndA=\iint_R F\left(\bar{r}(r,\theta)\right).N(r,\theta)\,drd\theta$$

$$¿\int_0^1\iint(-2r\cos\theta-5r\sin\theta+3r)\,drd\theta$$

$$¿\int_0^{2\pi}\left(\frac{-2}{2}r^2\cos\theta-\frac{5}{2}r^2+\frac{3}{2}r^2¿\right)¿ \text{ Limits for r = 0 and 1}$$

$$¿\int_0^{2\pi}\left(-\cos\theta-\frac{5}{2}\sin\theta+\frac{3}{2}\right)d\theta$$

$$¿\left(-\sin\theta+\frac{5}{2}\cos\theta+\frac{3}{2}\theta\right)Limits\,for\,\theta 0¿2\pi$$

$$=\left(0+\frac{5}{2}+3\pi\right)-\left(0+\frac{5}{2}+0\right)=3\pi$$

6) Verify Stokes's theorem for the vector $F=zi+xj+yk$ and the surface S represented by the come $z=\sqrt{x^2+y^2}$ where $0\le z\le 4$.

Let us proof that $\iint_R (curlF).ndA=\oint_c F.dr$

$r=[\cos s;\sin s;0]$ $r'=[-\sin s;\cos s;0]$ $F_1=zi$ $F_2=xj$ $F_3=yk$

$F(r(s))=[0,\cos s,\sin s]$

$$\oint_c F.dr=\int_0^{2\Pi} F(r(s).r'(s)ds$$

$$=\int_0^{2\Pi}[0+\cos s\times\cos s+0]ds$$

$$=\int_0^{2\Pi}\cos^2 sds$$

$$=\int_0^{2\Pi} ds+\int_0^{2\Pi}\cos 2sds$$

$$=s|_0^{2\Pi}+0$$

$$=2\Pi$$

We now consider the surface integral, we have $F_1=zi$ $F_2=xj$ $F_3=yk$ so can now calculate the $curlF$ .

$$curlF=\left(\frac{dy}{dy}-\frac{dx}{dz}\right)N_1+\left(\frac{dz}{dz}-\frac{dy}{dx}\right)N_2+\left(\frac{dx}{dx}-\frac{dz}{dy}\right)N_3$$

$$=curl(z,x,y)$$
$$=(1,1,1)$$

A normal vector of S is $N=grad(z-f(x,y))=[2x,2y,2z]$. Hence $(curlF)\cdot N=2x+2y+\frac{1}{16}$. Now $ndA=Ndxdy$ ,using polar coordinates $r,\theta$ defined by $x=r\cos\theta, y=r\sin\theta$ and denoting the projection of $S$ into the $xy-plane$ by R, we thus obtain:

$$\iint_R (curlF).ndA=\iint_R (curlF).Ndxdy$$

$$=\iint_R (x+y+\frac{1}{16}dxdy$$

$$=\int_0^{2\Pi}\int_0^4 (2r(\cos\theta+\sin\theta)+\frac{1}{16})rdrd\theta$$

$$=\int_0^{2\Pi} 32(\cos\theta+\sin\theta)+1\,d\theta$$

$$=0+\theta|_0^{2\Pi}$$

$$=2\Pi$$

Q7 Use the divergence theorem to find

$$\iint_S F.ndA$$

where $F=\left(x^2+sinyz\right)i+\left(y-xe^{-z}\right)j+z^{2k}$ and S is the surface od the region bounded by the cylinder: $x^2+y^2=4$ and the planes: $x+z=2, z=0$

Solution:

$$¿.\ V=2x+1+2z$$

$$\iiint_{000}^{2\pi 2 2-rcos\theta} 2rcos\theta+1+2z\,dz\,rdr\,d\theta$$

$$\iint_{00}^{2\pi 2}(2r¿¿2cos\theta z+rz¿+rz^2)_0^{2-rcos\theta}dr\,d\theta¿¿$$

$$\iint_{00}^{2\pi 2} 2r^2cos\theta(2-rcos\theta)+r(2-rcos\theta)+r(2-rcos\theta)^2\,dr\,d\theta$$

$$\iint_{00}^{2\pi 2}[4r¿¿2cos\theta-2r^3cos^2\theta+2r-r^2¿cos\theta+4r-4r^2cos\theta+r^2cos^2\theta]dr\,d\theta¿¿$$

$$\iint_{00}^{2\pi 2}[6r-2r^3cos^2\theta-r^2¿cos\theta+r^2cos^2\theta]dr\,d\theta¿$$

$$\int_0^{2\pi}[3r¿¿2¿\frac{-r^4}{2}cos^2\theta-\frac{r^3}{3}cos\theta+\frac{r^3}{3}cos^2\theta]_0^2\,d\theta¿¿$$

$$\int_0^{2\pi}[12-8cos^2\theta-\frac{8}{3}¿cos\theta+\frac{8}{3}cos^2\theta]\,d\theta¿$$

$$\int_0^{2\pi}[12-4(1+\cos 2\theta)-\frac{8}{3}¿cos\theta+\frac{4}{3}(1+\cos 2\theta)]\,d\theta¿$$

$$¿\left(12-\frac{8}{3}\theta-\frac{4}{3}\sin 2\theta-\frac{8}{3}sin\theta\right)_0^{2\pi}$$

$$¿\left(24\pi-\frac{16}{3}\pi-0-0\right)=\frac{56\pi}{3}$$